

# Median of 3 Permutations, 3-Cycles and 3-Hitting Set Problem

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IWOCA 2018, Singapore



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# Introduction



**Figure:** What activity would you like to do this weekend?



# Introduction



**Figure:** Children respond by ordering the 5 candidates activities



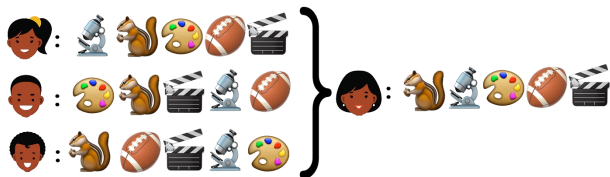
# Introduction



**Figure:** The mother finds a consensus using the median of permutation problem!



# Introduction - Model



$$\left. \begin{array}{l} \pi_1: [4, 5, 1, 2, 3] \\ \pi_2: [1, 5, 3, 4, 2] \\ \pi_3: [5, 2, 3, 4, 1] \end{array} \right\} \pi^*: [5, 4, 1, 2, 3]$$

$$m = 3, n = 5$$



# Kendall- $\tau$ distance and Kemeny score

## Kendall- $\tau$ distance

Given two permutations  $\pi, \sigma \in \mathcal{S}_n$ , this distance counts the number of order disagreements between pairs of elements:

$$Kt(\pi, \sigma) = \#\{(i, j) \mid i < j \text{ and } [(i \prec_{\pi} j \text{ and } j \prec_{\sigma} i) \text{ or } (j \prec_{\pi} i \text{ and } i \prec_{\sigma} j)]\}$$

## Kemeny Score

Given any set of permutations  $\mathcal{R} \subseteq \mathcal{S}_n$  and a permutation  $\pi \in \mathcal{S}_n$ , the *Kemeny score* is defined as:

$$K(\pi, \mathcal{R}) = \sum_{\sigma \in \mathcal{R}} Kt(\pi, \sigma)$$

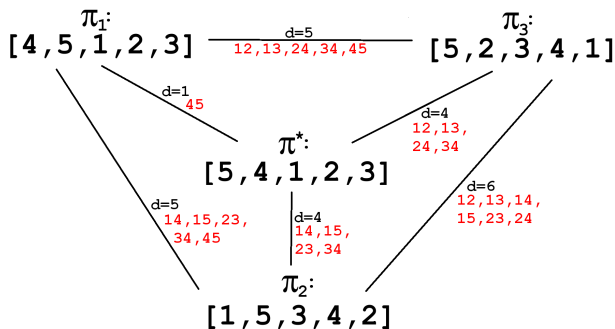


# Kemeny consensus

## The median permutation

Given  $\mathcal{R} \subseteq \mathcal{S}_n$ , we want to find a permutation  $\pi^* \in \mathcal{S}_n$  such that:

$$K(\pi^*, \mathcal{R}) \leq K(\pi, \mathcal{R}), \forall \pi \in \mathcal{S}_n.$$





# Quick overview of previous works

- Electoral System: "Kemeny-Young Method"

H.P. Young and A. Levenglick, *SIAM Journal on Applied Mathematics*, 1978

- Complexity: NP-Hard for  $m \geq 4$ ,  $m$  even and  $m \geq 7$ ,  $m$  odd, but  $m \in \{3, 5\}$  is unknown

C. Dwork and al. *Proceedings of the 10th WWW*, 2001

G. Bachmeier and al. *arXiv: <http://arxiv.org/abs/1704.06304v1>*, 2017

- Heuristics and Algorithms

F. Schalekamp and A. van Zuylen *Proceedings of the 11th SIAM ALENEX*, 2009

A. Ali and M. Meila *Mathematical Social Sciences*, 2012

- Space reduction techniques

M. Truchon *19th National Conference on AI*, 1998

G. Blin and al. *Pure Mathematics and Applications*, 2011

N. Betzler and al. *Autonomous Agents and Multi-Agent Systems*, 2014

R. Milosz and S. Hamel *Lecture Notes in Computer Science*, 2016

- Lower bounds

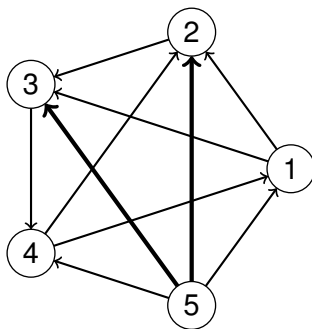
A. Davenport and J. Kalagnanam *19th National Conference on AI*, 2004

V. Conitzer and al. *21st National Conference on AI*, 2006



# Main result - Majority graph

$$\left. \begin{array}{l} \pi_1: [4, 5, 1, 2, 3] \\ \pi_2: [1, 5, 3, 4, 2] \\ \pi_3: [5, 2, 3, 4, 1] \end{array} \right\} \pi^*: [5, 4, 1, 2, 3]$$



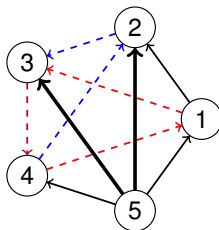
**Figure:** The majority graph of the set  $\mathcal{R} = \{[4, 5, 1, 2, 3], [1, 5, 3, 4, 2], [5, 2, 3, 4, 1]\}$ . Bold edges have weight 3 and thin edges have weight 1.



# Main result - 3-Cycle Theorem

## 3-Cycle Theorem

Let  $\mathcal{R} \subset \mathcal{S}_n$  be a set of 3 permutations. Let  $G_{\mathcal{R}} = (V, E)$  be its majority graph. Let  $\pi^*$  be any median permutation of  $\mathcal{R}$ . If an edge  $(i, j)$  of  $G_{\mathcal{R}}$  is not contained in any 3-cycle, then  $i \prec_{\pi^*} j$ .

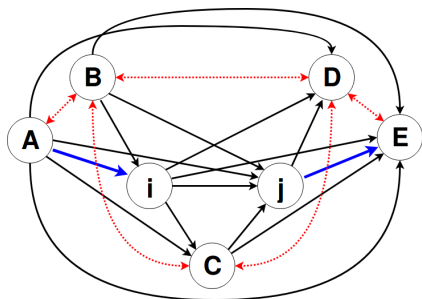


**Figure:** The majority graph of the set  $\mathcal{R} = \{[4, 5, 1, 2, 3], [1, 5, 3, 4, 2], [5, 2, 3, 4, 1]\}$ . Bold edges have weight 3 and thin edges have weight 1. The 3-cycles are shown with dashed edges.



# Main result - Sketch of proof

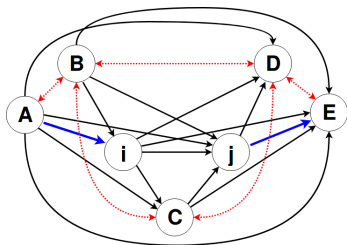
Let  $(i, j)$  be a pair of elements such that the corresponding edge is not contained in any 3-cycle. Now suppose by contradiction:  $\pi^* = \dots jKi \dots$   
We separate the set  $K$  of elements in 5 classes:



**Figure:** A black arrow is a preference relation, a blue bold arrow is an "Always" relation and a red two-sided dotted arrow is an unknown relation.



# Main result - Sketch of proof



$$\pi^* = \dots jKi \dots$$

## Step 1)

$$e, a \in K \text{ with } e \in E \text{ and } a \in A \implies e \not\prec_{\pi^*} a$$

$$\pi^* = \dots j \dots eSa \dots i \dots \text{ becomes } \pi^{*'} = \dots j \dots aeS \dots i \dots \text{ or } \pi^{*'} = \dots j \dots Sae \dots i \dots$$

## Step 2)

$$k \in K \implies k \notin A \cup E$$

$$\pi^* = \dots jSa \dots i \dots \text{ becomes } \pi^{*'} = \dots ajS \dots i \dots \text{ or } \pi^{*'} = \dots Saj \dots i \dots$$

## Step 3)

$K$  with only elements from  $B \cup C \cup D$ : swap  $i$  and  $j$

$$\pi^* = \dots jKi \dots \text{ becomes } \pi^{*'} = \dots iKj \dots$$



# Main result - Significance

## 3-Cycle Theorem

Let  $\mathcal{R} \subset \mathcal{S}_n$  be a set of 3 permutations. Let  $G_{\mathcal{R}} = (V, E)$  be its majority graph. Let  $\pi^*$  be any median permutation of  $\mathcal{R}$ . If an edge  $(i, j)$  of  $G_{\mathcal{R}}$  is not contained in any 3-cycle, then  $i \prec_{\pi^*} j$ .

## Significance of the 3-Cycle Theorem

- It includes all previous spaces reductions techniques for  $m = 3$  permutations.
- Improves lower bound results for  $m = 3$  permutations.
- When combined with an ILP solver (i.e. CPLEX), it improves the solving time for randomly generated data sets (1.6x, 48% pairs ordered) as well as for real data sets (3.7x, 87% pairs ordered) with  $m = 3$ .



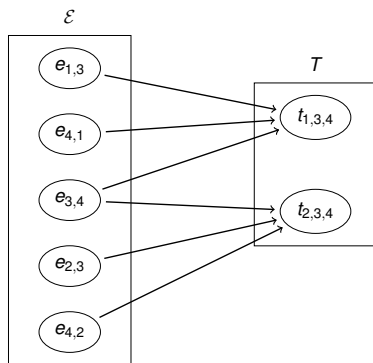
# Main result - 1st Conjecture

## 3-Cycle Conjecture for $m$ odd

Let  $\mathcal{R} \subset \mathcal{S}_n$  be a set of  $m$  permutations with  $m$  odd. Let  $G_{\mathcal{R}} = (V, E)$  be its majority graph. Let  $\pi^*$  be any median permutation of  $\mathcal{R}$ . If an edge  $(i, j)$  of  $G_{\mathcal{R}}$  is not contained in any 3-cycle, then  $i \prec_{\pi^*} j$ .



# Main result - 3-Hitting Set as a lower bound



**Figure:** 3-Hitting Set problem with  $\mathcal{E}$  as the set of involved edges  $I_E$  and  $\mathcal{T}$  the set of 3-cycles from the example. In this case, selecting edge  $e_{3,4}$  covers all 3-cycles thus  $S = \{e_{3,4}\}$  and  $\#S = 1$ . Reversing this edge in the original majority graph makes the majority graph acyclic.





# Main result - 2nd Conjecture

## 3-Hitting Set Conjecture for 3 permutations

Let  $\mathcal{R} \subset \mathcal{S}_n$  be a set of  $m$  permutations with  $m$  odd. Let  $G_{\mathcal{R}} = (V, E)$  be its majority graph. Let  $I_E$  be the set of edges involved in 3-cycles and  $T$  the set of 3-cycles. For any optimal solution  $S$  of the 3-Hitting Set problem, a median permutation can be constructed by reversing all edges  $(i, j) \in S$  in the majority graph and taking the topological ordering.

In short: **Solving the median of 3 permutations problem (M3P) amounts to solving a 3-Hitting Set problem (3HS)!**

Significance: ILP solving 19x faster on random data sets and 187x faster on real data sets (from PrefLib.org\*)

\*N. Mattei and T. Walsh, *Lecture Notes in Artificial Intelligence*, 2013



# Main result - 2nd Conjecture

## 3-Hitting Set Conjecture for 3 permutations

Let  $\mathcal{R} \subset \mathcal{S}_n$  be a set of  $m$  permutations with  $m$  odd. Let  $G_{\mathcal{R}} = (V, E)$  be its majority graph. Let  $I_E$  be the set of edges involved in 3-cycles and  $T$  the set of 3-cycles. For any optimal solution  $S$  of the 3-Hitting Set problem, a median permutation can be constructed by reversing all edges  $(i, j) \in S$  in the majority graph and taking the topological ordering.

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# Conclusion

- The median of permutations is an NP-Hard, but the  $m = 3$  case is an open question
- The 3-Cycle theorem is a strong space reduction technique
- We conjecture that it holds for  $m$  odd
- The 3HS gives a very tight lower bound
- We conjecture that solving the M3P amounts to solving a 3HS



# Questions?

Thanks!



# Selected references

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